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# Medial layer graphs of equivelar 4-polytopes

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**Abstract**

In any abstract 4-polytope $\mathcal{P}$, the faces of ranks 1 and 2 constitute, in a natural way, the vertices of a medial layer graph $\mathcal{G}$. We prove that when $\mathcal{P}$ is finite, self-dual and regular (or chiral) of type $\{3, q, 3\}$, then the graph $\mathcal{G}$ is finite, trivalent, connected and 3-transitive (or 2-transitive). Given such a graph, a reverse construction yields a poset with some structure (a polystroma); and from a few well-known symmetric graphs we actually construct new 4-polytopes. As a by-product, any such 2- or 3-transitive graph yields at least a regular map (i.e. 3-polytope) of type $\{3, q\}$.


*MSC:* primary 05C25; secondary 51M20

## 1. Symmetric trivalent graphs

We begin by briefly outlining some basic ideas concerning symmetric graphs [3, chapter 18–19]. Although many of the following results generalize to graphs of higher valency, we shall for brevity *assume outright that $\mathcal{G}$ is a simple, finite, connected trivalent graph* (so that each vertex has valency 3).

By a *$t$-arc* in $\mathcal{G}$ we mean a list of vertices $[v] = [v_0, v_1, \ldots, v_t]$ such that $\{v_{i-1}, v_i\}$ is an edge for $1 \le i \le t$, but no $v_{i-1} = v_{i+1}$. When $t \ge 1$, a *successor* of the $t$-arc $[v]$ is a $t$-arc of the form $[v^{(k)}] := [v_1, \ldots, v_t, y_k]$. Clearly, $[v]$ has two successors $[v^{(1)}]$ and $[v^{(2)}]$, taking $v_{t-1}, y_1, y_2$ to be the vertices adjacent to $v_t$ (see Fig. 1).

Tutte has shown that there exists a maximal value of $t$ such that the automorphism group $\mathrm{Aut}(\mathcal{G})$ is transitive on $t$-arcs, and indeed, for any graph $\mathcal{G}$ we must have $t \le 5$ ([3, Theorem 18.6]). Tutte also proved that, for $t \ge 1$, $\mathrm{Aut}(\mathcal{G})$ is transitive on $t$-arcs if and only if for

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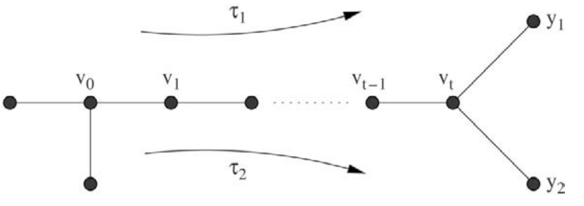


Fig. 1. Successors and shunts.

some $t$-arc $[v]$ there exist special automorphisms $\tau_1, \tau_2$, which we shall call *shunts*, such that $[v]\tau_k = [v^{(k)}]$ ([24, 7.54]). Such arc-transitive graphs are said to be *symmetric*.

Still taking $t \geq 1$, we say that $\mathcal{G}$ is *$t$-transitive* if $\mathrm{Aut}(\mathcal{G})$ is transitive on $t$-arcs, but not on $(t+1)$-arcs in $\mathcal{G}$. For a fixed $t$-arc $[v]$, the *stabilizer sequence* for $[v]$ is

$$\mathrm{Aut}\,\mathcal{G} \supset B_t \supset B_{t-1} \supset \cdots \supset B_1 \supset B_0,$$

where the subgroup $B_j$ is the pointwise stabilizer of $\{v_0, \ldots, v_{t-j}\}$.

Suppose then that $\mathcal{G}$ is $t$-transitive, for some $t \geq 1$ (so that also $t \leq 5$). Since $\mathrm{Aut}(\mathcal{G})$ is transitive on $r$-arcs, for $r \leq t$, the subgroup $B_j$ is conjugate to that obtained from any other $t$-arc. In particular, $B_t$ is the vertex stabilizer, whereas $B_0$ is the pointwise stabilizer of the whole arc. In fact, $B_0 = \{e\}$ is trivial ([3, Proposition 18.1]), so that $\mathrm{Aut}(\mathcal{G})$ acts sharply transitively on $t$-arcs. It follows that the shunts $\tau_k$ are uniquely defined for a given $t$-arc $[v]$.

As preparation for a more refined description of $\mathrm{Aut}(\mathcal{G})$, consider next the unique automorphism $\alpha$ which reverses the basic $t$-arc $[v]$. Then $\alpha$ has period 2 and $\alpha\tau_1\alpha$ is either $\tau_1^{-1}$ or $\tau_2^{-1}$. We shall say that $\mathcal{G}$ is of *type* $t^+$ or $t^-$, respectively. The actual possibilities appear in Theorem 1(d) below, along with several other remarkable results concerning $\mathrm{Aut}(\mathcal{G})$ (see [3, chapter 18]):

**Theorem 1.** *Suppose $\mathcal{G}$ is a finite connected $t$-transitive trivalent graph, with $1 \leq t$, and suppose $\mathcal{G}$ has $N$ vertices. Then*

(a) *We have these group orders:*

$$|B_j| = 2^j \qquad 0 \leq j \leq t-1.$$
$$|B_t| = 3 \cdot 2^{t-1}.$$
$$|\mathrm{Aut}(\mathcal{G})| = 3 \cdot N \cdot 2^{t-1}.$$

(b) *If* $t \geq 2$, $\mathrm{Aut}(\mathcal{G})$ *is generated by* $\tau_1, \tau_2$.

(c) *The stabilizers $B_j$ are determined up to isomorphism by $t$:*

| $t$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| 1 | $\mathbb{Z}_3$ | | | | |
| 2 | $\mathbb{Z}_2$ | $\mathbb{S}_3$ | | | |
| 3 | $\mathbb{Z}_2$ | $(\mathbb{Z}_2)^2$ | $\mathbb{D}_{12}$ | | |
| 4 | $\mathbb{Z}_2$ | $(\mathbb{Z}_2)^2$ | $\mathbb{D}_8$ | $\mathbb{S}_4$ | |
| 5 | $\mathbb{Z}_2$ | $(\mathbb{Z}_2)^2$ | $(\mathbb{Z}_2)^3$ | $\mathbb{D}_8 \times \mathbb{Z}_2$ | $\mathbb{S}_4 \times \mathbb{Z}_2$ |

(d) *$\mathcal{G}$ is one of 7 types:* $1^-, 2^+, 2^-, 3^+, 4^+, 4^-$ *or* $5^+$.

(Here $\mathbb{Z}_k$ is the cyclic group of order $k$, $\mathbb{D}_{2k}$ is the dihedral group of order $2k$, $\mathbb{S}_k$ is the symmetric group of degree $k$.)

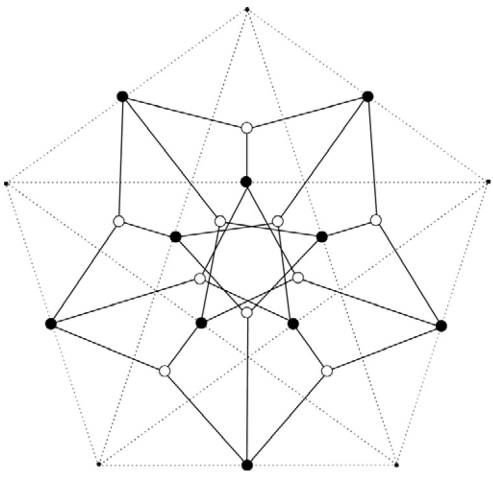

Fig. 2. The Levi graph for the Desargues configuration $10_3$.

A nearly complete list of symmetric trivalent graphs, with at most 512 vertices, was compiled by R.M. Foster over several decades of the last century. An enhanced version of this *Census*, with detailed descriptions of the individual graphs, appears in [5]. We refer to [19] for historical details and to [7] for a recent update. Let us now consider a pertinent example:

**Example 1.** The graph $\mathcal{G} = \mathbf{20B}$ from *Foster's Census*.

We display $\mathcal{G}$ in Fig. 2, along with some underlying scaffolding. The slightly fainter regular pentagon and pentagram arise as the most symmetric 2-dimensional projection of the regular 4-simplex $\mathcal{P} = \{3, 3, 3\}$ in $\mathbb{R}^4$. Observe that each black node in $\mathcal{G}$ indicates the midpoint of an edge in the simplex. Each such edge is complementary to a triangular face whose centroid gives a white node in $\mathcal{G}$. Naturally, a black node and white node are adjacent in $\mathcal{G}$ just when the corresponding edge and triangle are incident in $\mathcal{P}$. Since each triangular face has 3 edges and dually each edge lies on 3 triangles, we confirm that $\mathcal{G}$ is trivalent and bipartite.

Anticipating the general construction described in the abstract setting in Section 2 below, we say that $\mathcal{G}$ is the medial layer graph for the polytope $\mathcal{P}$. Looking further ahead to Theorem 2, we know that $\mathcal{G}$ must be 3-transitive, since $\mathcal{P}$ is a self-dual regular convex polytope. However, here we can verify this fact directly, noting first that each of the 240 symmetries and dualities of $\mathcal{P}$ induces an automorphism of $\mathcal{G}$. Since $\mathcal{G}$ has two distinct kinds of 4-arcs, $\mathrm{Aut}(\mathcal{G})$ must have order $240 = 20 \cdot 3 \cdot 2^{3-1}$; and $\mathcal{G}$ must be 3-transitive.

If we extend the 10 edges and 10 triangular faces of $\mathcal{P}$ to their affine hulls and intersect these with a generic hyperplane in $\mathbb{R}^4$, we obtain a (3-dimensional) Desargues configuration $10_3$. In this configuration, the 10 new points lie by threes on the 10 new lines, with three lines through each point. Clearly, $\mathcal{G}$ can be viewed as the *Levi graph* (indicating incidences) for the Desargues configuration.

Notice that complementary pairs of black and white nodes are antipodal and at distance 5 in $\mathcal{G}$. If we identify such pairs, we obtain the (non-bipartite) Petersen graph (**10** in the *Census*). In the other direction, note that $\mathcal{G}$ is the canonical double covering of the Petersen graph [3, 19a-b]. □

To summarize, we have seen that the graph **20B** has a natural Euclidean realization as the medial layer graph of the self-dual regular convex polytope $\{3, 3, 3\}$. The only other such convex 4-polytope is the 24 cell $\{3, 4, 3\}$; and from it we likewise obtain a 3-transitive graph on 192 vertices (graph **192A** in the *Census*). To expand our list of examples we must leave convexity and investigate the more general combinatorial setting of abstract polytopes.

## 2. Abstract 4-polytopes and their medial layer graphs

The sort of graphs which we investigate in this paper are constructed from the 1- and 2-faces in abstract regular, or chiral, 4-polytopes. Putting aside symmetry for the moment, we recall that an *(abstract) n-polytope* $\mathcal{P}$ is a partially ordered set with a strictly monotone rank function having range $\{-1, 0, \ldots, n\}$. An element $F \in \mathcal{P}$ with $\operatorname{rank}(F) = j$ is called a *j-face*; typically $F_j$ will indicate a $j$-face. We require that $\mathcal{P}$ have two improper faces: the unique least face $F_{-1}$ and the unique greatest face $F_n$. Furthermore, each maximal chain or *flag* in $\mathcal{P}$ must contain $n+2$ faces. Next, $\mathcal{P}$ must satisfy a *homogeneity property*: whenever $F < G$ with $\operatorname{rank}(F) = j-1$ and $\operatorname{rank}(G) = j+1$, there are exactly two $j$-faces $H$ with $F < H < G$. It follows that for $0 \le j \le n-1$ and any flag $\Phi$, there exists a unique *adjacent* flag $\Phi^j$, differing from $\Phi$ in just the rank $j$ face. With this notion of adjacency the flags of $\mathcal{P}$ form a *flag graph* (not to be confused with the medial layer graphs appearing below).

The final defining property of $\mathcal{P}$ is that it be *strongly flag-connected*. This means that the flag graph for each section is connected. (Whenever $F \le G$ are faces of ranks $j \le k$ in $\mathcal{P}$, the *section* $G/F := \{H \in \mathcal{P} | F \le H \le G\}$ is in its own right a polytope of rank $k-j-1$; see [17, 2A] for details.)

Since our main concern is with 4-polytopes (and their sections), we now tailor our discussion to this case. A (rank 4) polytope $\mathcal{P}$ is said to be *equivelar* of type $\{p_1, p_2, p_3\}$ if, for $j = 1, 2, 3$, whenever $F$ and $G$ are incident faces of $\mathcal{P}$ with $\operatorname{rank}(F) = j-2$ and $\operatorname{rank}(G) = j+1$, then the rank 2 section $G/F$ has the structure of a $p_j$-gon (independent of the choice of $F < G$). Thus, each 2-face (polygon) of $\mathcal{P}$ is isomorphic to a $p_1$-gon, and there are $p_3$ of these arranged around each 1-face (edge) of $\mathcal{P}$; and in every 3-face (facet) of $\mathcal{P}$, each 0-face is surrounded by an alternating cycle of $p_2$ edges and $p_2$ polygons.

This sort of local combinatorial uniformity does not necessarily imply any non-trivial global symmetry in $\mathcal{P}$. For that we must consider the automorphism group $\operatorname{Aut}(\mathcal{P})$ consisting of all order preserving bijections $\alpha$ on $\mathcal{P}$. Also, if $\mathcal{P}$ admits a duality $\delta$ (order reversing bijection), then $\mathcal{P}$ is said to be *self-dual*; and clearly then $\operatorname{Aut}(\mathcal{P})$ has index 2 in the group $D(\mathcal{P})$ of all automorphisms and dualities. (It will be useful to simply let $D(\mathcal{P}) = \operatorname{Aut}(\mathcal{P})$ when $\mathcal{P}$ is not self-dual.) If $\mathcal{P}$ is self-dual and equivelar, then it has type $\{p_1, p_2, p_1\}$.

**Definition 1.** Let $\mathcal{P}$ be a 4-polytope. The associated *medial layer graph* $\mathcal{G}(\mathcal{P})$, or briefly $\mathcal{G}$, is the simple graph whose vertex set is comprised of all 1-faces and 2-faces in $\mathcal{P}$, two such taken to be adjacent when incident in $\mathcal{P}$.

**Remark and thanks:** For smoother terminology, we often simply write *medial graph*, recognizing, however, that this term already has a somewhat different meaning in topological graph theory (see [1], for example). The authors here wish to thank the referees for pointing this out, and for many other useful suggestions.

Note that a medial graph $\mathcal{G}$ is bipartite. We shall say that a $t$-arc in $\mathcal{G}$ is of *type* 1 (resp. *type* 2) if its initial vertex is a 1-face (resp. 2-face) of $\mathcal{P}$. It follows easily from the flag connectedness of $\mathcal{P}$ that $\mathcal{G}$ is connected as a graph. Note that if $\mathcal{P}$ is equivelar of type $\{p_1, p_2, p_3\}$, then $\mathcal{G}$ has alternate $p_1$-valent and $p_3$-valent vertices, situated along cycles of length $2p_2$.

In Fig. 3 we show a fragment of a polytope $\mathcal{P}$ of type $\{3, 6, 3\}$. The vertices of $\mathcal{G}$ are here represented as black and white discs (at what may seem to be “centroids” of the corresponding

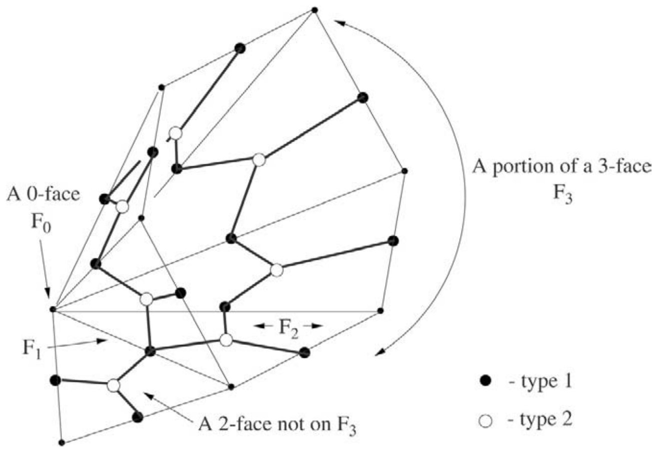


Fig. 3. A fragment of a polytope of type {3, 6, 3}.

1-faces and 2-faces of $\mathcal{P}$); and the edges of $\mathcal{G}$ are indicated by heavy lines. (The diagram is useful enough, even though $\mathcal{P}$ might actually have no such easily visualized representation.) Families of such graphs, generally quite symmetric, have been examined in [8,11,25] and [26], for example.

Clearly the action of $D(\mathcal{P})$ on $\mathcal{G}$ induces a homomorphism

$$D(\mathcal{P}) \longrightarrow \mathrm{Aut}(\mathcal{G}).$$

In fact, if $\mathcal{P}$ is equivelar of type $\{p_1, p_2, p_3\}$, with $p_1 \geq 3$ and $p_3 \geq 3$ (as we shall generally assume), then this action is faithful. To see this, suppose that $\eta \in D(\mathcal{P})$ fixes each vertex of $\mathcal{G}$, namely all 1- and 2-faces of $\mathcal{P}$. On any particular 2-face (i.e. polygon in $\mathcal{P}$), $\eta$ acts as an automorphism fixing each 1-face (i.e. edge of this polygon). Since $p_1 \geq 3$, $\eta$ must fix each 0-face (i.e. vertex) of this particular polygon. In short, $\eta$ fixes all 0-faces of $\mathcal{P}$; since $p_3 \geq 3$, a dual argument shows that $\eta$ fixes all 3-faces of $\mathcal{P}$. Thus $\eta = e$. (We require $p_1 \geq 3$, for example, since the digon {2} does have an automorphism fixing the two 1-faces but flipping the two 0-faces.)

It follows that we may regard $D(\mathcal{P})$, or $\mathrm{Aut}(\mathcal{P})$, as a subgroup of $\mathrm{Aut}(\mathcal{G})$. However, it can happen that $D(\mathcal{P})$ is a proper subgroup of $\mathrm{Aut}(\mathcal{G})$; see, for example, the construction of the Gray graph in [20].

We turn now to two significant classes of highly symmetric polytopes.

## 3. Medial layer graphs of regular self-dual 4-polytopes of type {3, $q$, 3}

An $n$-polytope $\mathcal{P}$ is called *regular* if $\mathrm{Aut}(\mathcal{P})$ is transitive on the flags of $\mathcal{P}$. Since $\mathrm{Aut}(\mathcal{P})$ acts freely on flags [17, Proposition 2A4], these are in a sense the most symmetric of polytopes. We now review some key constructions contained in [17, 2B–2E].

Assuming again that $n = 4$, we fix a base flag $\Phi = \{F_{-1}, F_0, F_1, F_2, F_3, F_4\}$ in the regular polytope $\mathcal{P}$. For $0 \leq j \leq 3$, let $\rho_j$ be the (unique) automorphism with $(\Phi)\rho_j = \Phi^j$ (the flag adjacent to $\Phi$ at rank $j$). Then $\mathrm{Aut}(\mathcal{P})$ is generated by $\rho_0, \rho_1, \rho_2, \rho_3$, which actually are involutions satisfying at least the relations

$$\begin{aligned} &\rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_1\rho_3)^2 = e \\ &(\rho_0\rho_1)^{p_1} = (\rho_1\rho_2)^{p_2} = (\rho_2\rho_3)^{p_3} = e, \end{aligned} \tag{1}$$

with $2 \le p_1, p_2, p_3 \le \infty$. Indeed, $\mathcal{P}$ is equivelar of type $\{p_1, p_2, p_3\}$. Furthermore, an *intersection condition* on standard subgroups holds:

$$\langle \rho_i | i \in I \rangle \cap \langle \rho_i | i \in J \rangle = \langle \rho_i | i \in I \cap J \rangle \tag{2}$$

for all $I, J \subseteq \{0, 1, 2, 3\}$. Any group, which like $\mathrm{Aut}(\mathcal{P})$ is generated by specified involutions satisfying (1) and (2), is called a *string C-group* (here of rank 4). (Such a group is a special quotient of a Coxeter group with string diagram.)

Conversely, given any string $C$-group $\Gamma = \langle \rho_0, \ldots, \rho_3 \rangle$, one may construct a regular 4-polytope $\mathcal{P} = \mathrm{Pol}(\Gamma)$, of type $\{p_1, p_2, p_3\}$, with $\mathrm{Aut}(\mathrm{Pol}(\Gamma)) = \Gamma$:

**Definition 2.** For any (rank 4) string $C$-group $\Gamma = \langle \rho_0, \ldots, \rho_3 \rangle$, let $\Gamma_{-1} := \Gamma$, $\Gamma_0 := \langle \rho_1, \rho_2, \rho_3 \rangle$, $\Gamma_1 := \langle \rho_0, \rho_2, \rho_3 \rangle$, $\Gamma_2 := \langle \rho_0, \rho_1, \rho_3 \rangle$, $\Gamma_3 := \langle \rho_0, \rho_1, \rho_2 \rangle$, and $\Gamma_4 := \Gamma$. Then the $j$-faces of the 4-polytope $\mathrm{Pol}(\Gamma)$ are all cosets

$$\Gamma_j \varphi, \varphi \in \Gamma, -1 \le j \le 4,$$

where $\Gamma_j \varphi \le \Gamma_k \tau$ if and only if

$$-1 \le j \le k \le 4 \qquad \text{and} \qquad \Gamma_j \varphi \cap \Gamma_k \tau \ne \emptyset.$$

**Remarks.** We also have $\mathrm{Pol}(\mathrm{Aut}(\mathcal{P})) \cong \mathcal{P}$. Thus, in each rank, the regular polytopes correspond exactly to the string $C$-groups (see [17, 2E]). For readers acquainted with diagram geometries, we note that a regular polytope is a thin, residually connected geometry with string diagram [4, Section 3.4].

Any Coxeter group $\Gamma$ with a string diagram is certainly a string $C$-group [17, Proposition 3D3], although $\Gamma$ and the corresponding polytope $\mathcal{P}$ may well be infinite. In particular, switching to rank 2, we note that the automorphism group of the regular polygon $\{p\}$, $p \in \{2, 3, \ldots, \infty\}$, is the dihedral group $\mathbb{D}_{2p}$ (or $\mathbb{D}_\infty$). □

Returning to the regular 4-polytope $\mathcal{P}$, we observe that $\mathcal{P}$ is self-dual if and only if $\mathrm{Aut}(\mathcal{P})$ admits an involutory group automorphism which transposes $\rho_0, \rho_3$ and $\rho_1, \rho_2$. Equivalently, there exists an involutory duality, or *polarity* $\delta \in D(\mathcal{P})$, which reverses the basic flag $\Phi$. Then $\delta^2 = e$, $\delta\rho_0\delta = \rho_3$, $\delta\rho_1\delta = \rho_2$; and $D(\mathcal{P}) \simeq \mathrm{Aut}(\mathcal{P}) \rtimes \mathbb{Z}_2$ (see [17, 2B17 and 2E12]).

For the remainder of this section, we shall assume that $\mathcal{P}$ is regular and self-dual with $p_1 = p_3 = 3$ and $q := p_2 \ge 2$. The medial graph $\mathcal{G} = \mathcal{G}(\mathcal{P})$ is therefore a rather symmetric trivalent graph.

As a convenient notation we now define $v_1 := F_1, v_2 := F_2$, and in general let $v_0 = v_{2q}, v_1, v_2, \ldots, v_{2q-1} = v_{-1}$ denote alternate edges and triangles in the rank 2 section $F_3/F_0$ of $\mathcal{P}$. Thus each $v_j$ is adjacent in $\mathcal{G}$ to $v_{j\pm1}$, taking subscripts mod $2q$. We also let $w_j$ be the third vertex adjacent to $v_j$ in $\mathcal{G}$, and take $x := (v_{-1})\rho_3$ and $y$ to be the two other vertices adjacent to $w_1$, and likewise let $s := (v_4)\rho_0$ and $t$ be the two others adjacent to $w_2$ (see Fig. 4).

An explicit calculation shows that the $\rho_j, \delta$ act on the vertices of $\mathcal{G}$ as follows:

$$\begin{aligned}
\rho_0 &= (v_0)(v_1)(v_2)(w_1)(v_{-1}w_0)(v_3w_2)(xy)(v_4s)\cdots \\
\rho_1 &= (v_2)(w_2)(s)(t)(v_1v_3)(v_0v_4)(w_1w_3)\cdots \\
\rho_2 &= (v_1)(w_1)(x)(y)(v_{-1}v_3)(v_0v_2)(w_0w_2)\cdots \\
\rho_3 &= (v_1)(v_2)(v_3)(w_2)(v_0w_1)(v_4w_3)(w_0y)(v_{-1}x)\cdots \\
\delta &= (v_1v_2)(v_0v_3)(v_{-1}v_4)(w_0w_3)\cdots.
\end{aligned} \tag{3}$$

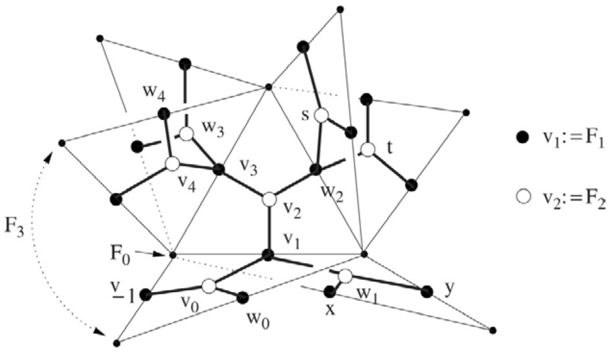


Fig. 4. Another view of a polytope of type $\{3, q, 3\}$.

Since the dualities

$$\begin{aligned} \tau_1 &:= \rho_2\delta = (\ldots v_0 v_1 v_2 v_3 v_4 \ldots)\ldots \\ \tau_2 &:= \rho_0\rho_2\delta = (\ldots v_0 v_1 v_2 v_3 w_3 \ldots)\ldots \end{aligned} \tag{4}$$

act as shunts on the 3-arc $[v] = [v_0, v_1, v_2, v_3]$, we conclude that $\mathrm{Aut}(\mathcal{G})$ is transitive on at least 3-arcs. (Even when $\mathcal{P}$ is not self-dual, it is easy to check that $\mathrm{Aut}(\mathcal{G})$ is separately transitive on 3-arcs *of each type*.) We now find by routine calculation that

$$\begin{aligned} &\delta\tau_1\delta = \tau_1^{-1}, &&\delta\tau_2\delta = \tau_2^{-1} \\ &\rho_0 = \tau_2\tau_1^{-1}, &&\rho_3 = \tau_2^{-1}\tau_1 \\ &\rho_1 = \delta\tau_1, &&\rho_2 = \tau_1\delta, \end{aligned} \tag{5}$$

so that $D(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2, \rho_3, \delta\rangle = \langle \tau_1, \tau_2, \delta\rangle$.

**Theorem 2.** *Suppose $\mathcal{P}$ is a finite, regular self-dual polytope of type $\{3, q, 3\}$. Then the medial layer graph $\mathcal{G}$ is bipartite, trivalent and 3-transitive, with $\mathrm{Aut}(\mathcal{G}) \simeq D(\mathcal{P})$.*

**Proof.** By our remarks in Section 1 we know that $\mathcal{G}$ is $t$-transitive for $t = 3, 4$ or $5$. If $t = 3$, we have by Theorem 1(b) that $\mathrm{Aut}(\mathcal{G}) = \langle \tau_1, \tau_2\rangle$. But recall that $D(\mathcal{P})$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathcal{G})$. Thus $\delta$ is superfluous and $\mathrm{Aut}(\mathcal{G}) \simeq D(\mathcal{P})$. It remains to rule out the cases $t = 4, 5$.

When $t = 4$, the vertex stabilizer $B_4 \simeq \mathbb{S}_4$ (Theorem 1(c)), which contains no element of order 6. However, $\rho_0\rho_2\rho_3$ has order 6 and stabilizes $v_1$.

In order to exclude the case $t = 5$ we must take a deeper look at the structure of $\mathrm{Aut}(\mathcal{G})$. It follows from the considerations in [6, Section 1] that, for each of the 7 types in Theorem 1(d), the group $\mathrm{Aut}(\mathcal{G})$ for a $t$-transitive graph $\mathcal{G}$ has generators $h, a, p$ which satisfy certain universal relations. Furthermore, $\mathrm{Aut}(\mathcal{G})$ possesses a subgroup $H$ ($\simeq B_t$, and certainly depending on $t$) whose right cosets provide, in a natural way, the vertices of a graph isomorphic to $\mathcal{G}$. Indeed, in this new description, the 'base vertex' $H$ is adjacent to $Ha$, $Hah$, $Hah^{-1}$; and we can reconstruct $\mathcal{G}$ from the natural action of $\mathrm{Aut}(\mathcal{G})$ on these cosets. (See Fig. 5. For other approaches, consider [2] or [12].)

In particular, for type $5^+$ we have

$$H = \langle h, p, apa, ah^{-1}apaha\rangle \simeq \mathbb{S}_4 \times \mathbb{Z}_2.$$

Now it is easy to check that the six vertices at distance two from vertex $H$ are $Haha$, $Hahah$, $Hahah^{-1}$ and $Hah^{-1}a$, $Hah^{-1}ah$, $Hah^{-1}ah^{-1}$. Also, up to conjugacy, the unique element of

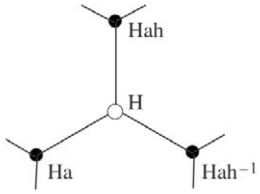


Fig. 5. Local structure in a $t$-transitive graph.

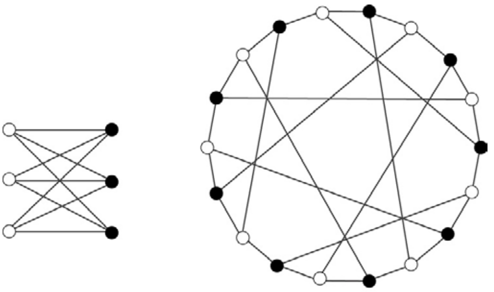


Fig. 6. (a) $K_{3,3}$, and (b) the Levi graph for the Pappus configuration $9_3$.

order 6 in $H$ is $hp$, which (because of the universal relations) permutes these six vertices in the two cycles of 3 indicated just above. Now compare the polytopal case, in which the required element of order 6 is conjugate to

$$\rho_0\rho_2\rho_3 = (x w_0 v_3 y v_{-1} w_2)\ldots,$$

which permutes the six vertices in a single cycle. (Since the girth of a 5-transitive graph is at least 8 [3, Proposition 17.1], there can be no collapse in this 6-cycle.) Thus $\mathcal{G}$ cannot be 5-transitive. □

Let us consider the two smallest graphs arising from the theorem:

**Example 2.** Graphs **6** and **18** in the *Census*.

The Thomsen graph $K_{3,3}$ (sometimes called Tutte's 4-cage) is the medial graph of the spherical honeycomb $\{3, 2, 3\}$; see Fig. 6(a). This graph clearly has $2 \cdot (3!)^2 = 72 = 6 \cdot 3 \cdot 2^{3-1}$ automorphisms, so that it must indeed be 3-transitive. (We refer to [8, p. 309] for a short description of the honeycomb, which has three faces of each rank $j = 0, 1, 2, 3$.)

Fig. 6(b) shows the Levi graph of the (self-dual) Pappus configuration $9_3$. This 3-transitive graph on 18 nodes is the only symmetric graph with that many nodes. Comparing group orders, we conclude that it must also be the medial graph of the universal polytope of type $\{\{3, 6\}_{(1,1)}, \{6, 3\}_{(1,1)}\}$ (see [17, p. 421]). Again, the polytope has only three 0-faces and three 3-faces. □

## 4. From 3-transitive graphs to regular polytopes

We next examine the possibility of a converse construction: given a suitable graph $\mathcal{G}$, can we reconstruct a (regular and self-dual) polytope $\mathcal{P}$ whose medial graph is $\mathcal{G}$? Let us suppose then that $\mathcal{G}$ is a finite, connected, trivalent and 3-transitive graph. However, for future purposes, we do not now assume that $\mathcal{G}$ is bipartite.

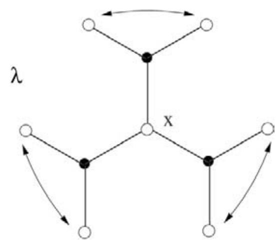


Fig. 7. The unique involution fixing a vertex and its neighbours.

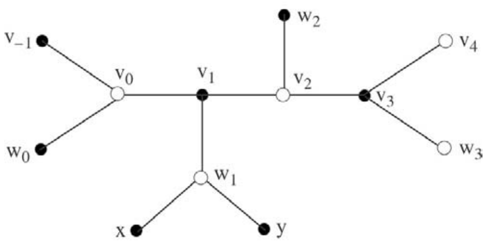


Fig. 8. Defining the key generators.

We begin by recording some simple yet very useful consequences of the fact that Aut($\mathcal{G}$) is sharply transitive on 3-arcs in $\mathcal{G}$. First note that if an automorphism $\lambda \neq e$ fixes a vertex $x$ and two neighbours, then $\lambda$ must fix the third neighbour and swap in pairs those vertices at distance two from $x$ (and from each other) (see Fig. 7).

Indeed, $\lambda$ has period 2. (The six vertices at distance 2 are distinct unless $\mathcal{G} \simeq K_{3,3}$.)

Now let $[v] = [v_0, v_1, v_2, v_3]$ be a fixed 3-arc, and let $w_1, w_2$ be the third vertices adjacent to $v_1, v_2$.

Motivated by (3), we can now make the key

**Definition 3.** Suppose that $\mathcal{G}$ is a finite, connected, trivalent and 3-transitive graph. With vertices labelled as in Fig. 8, let $\rho_0$ to be the unique non-trivial automorphism fixing $v_1$ and its neighbours; similarly define $\rho_1$ at $w_2$, $\rho_2$ at $w_1$ and $\rho_3$ at $v_2$. Let $\Gamma = \Gamma(\mathcal{G}) := \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$.

Note that each $\rho_j$ must have period 2. Still motivated by the previous section, we further let $\delta$ be the (unique) automorphism reversing $[v]$. Note that $\delta$ must have period 2. (As we shall soon see, in some cases $\delta$ can be identified with the polarity of Section 3.) Next let $\tau_1, \tau_2$ be the generating shunts, which shift the vertices of $[v]$ one step. Observe that $\tau_j \delta = (v_1)(v_0 v_2) \ldots$ must fix $w_1$ and either fix or flip the other two neighbours $x, y$ of $w_1$. Now if $(x)\tau_1\delta = (x)\tau_2\delta$, then $\tau_1\delta = \tau_2\delta$ by agreement on the 3-arc $[v_0, v_1, w_1, x]$. Since $\tau_1 \neq \tau_2$, this implies that exactly one of the $\tau_j\delta$, say $\tau_1\delta$, fixes both $x$ and $y$. Thus $\tau_1\delta$ is an involution and in fact equals $\rho_2$, so that $\delta\tau_1\delta = \tau_1^{-1}$. Likewise, since $\tau_1^{-1} \neq \tau_2^{-1}$, we must also have $\delta\tau_2\delta = \tau_2^{-1}$. (This proves that Aut($\mathcal{G}$) is always of type $3^+$, as indicated in Theorem 1(d); see [3, p. 145].)

We continue to label vertices by setting $v_4 := (v_3)\tau_1$, so that $w_3 := (v_3)\tau_2$ is the third vertex adjacent to $v_3$. Thus

$$\begin{aligned} \delta &= (v_0 v_3)(v_1 v_2)(w_1 w_2) \ldots \\ \tau_1 &= (\ldots v_0 v_1 v_2 v_3 v_4 \ldots) \ldots \\ \tau_2 &= (\ldots v_0 v_1 v_2 v_3 w_3 \ldots) \ldots . \end{aligned} \tag{6}$$

We record several useful properties of these automorphisms in

**Lemma 1.** *Let $\Gamma = \Gamma(\mathcal{G}) = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ be the group constructed in Definition 3. Then*

(a) $\delta^2 = \rho_j^2 = e$, *and* $\delta \rho_j \delta = \rho_{3-j}$, $0 \le j \le 3$.
(b) $\Gamma$ *has index* $k \le 2$ *in* $\mathrm{Aut}(\mathcal{G})$.
(c) $(\rho_0\rho_1)^3 = (\rho_1\rho_2)^q = (\rho_2\rho_3)^3 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_1\rho_3)^2 = e$, *where* $q \ge 2$ *is the period of* $\tau_1^2$.
(d) $\rho_0 = \tau_1\tau_2^{-1} = \tau_2\tau_1^{-1}$; $\rho_1 = \delta\tau_1 = \tau_1^{-1}\delta$;
$\rho_2 = \tau_1\delta = \delta\tau_1^{-1}$; $\rho_3 = \tau_1^{-1}\tau_2 = \tau_2^{-1}\tau_1$.

**Proof.** Verification is routine. For example, $\tau_1\tau_2^{-1} \neq e$ fixes $v_0, v_1, v_2$ hence equals $\rho_0$. Similarly, since $(v_1)\delta = v_2$, we have $\delta\rho_0\delta = \rho_3$.

Note that $\rho_0\rho_2 \neq e$ fixes *adjacent* vertices $v_1, w_1$, and their neighbours, and so must have period 2. Similarly, consider $\rho_0$ and $\rho_1$, which fix vertices $v_1, w_2$ separated by distance 2. Then $\rho_0\rho_1\rho_0$ and $\rho_1\rho_0\rho_1$ each fix $v_3 = (w_2)\rho_0 = (v_1)\rho_1$ and its neighbours. Thus $\rho_0\rho_1\rho_0 = \rho_1\rho_0\rho_1$. Likewise, because $w_1$ and $v_2$ are at distance 2, we have $(\rho_2\rho_3)^3 = e$. Finally, (b) follows from (d) and (a). □

**Remark.** We can say a little more if $\mathcal{G}$ is bipartite. Since $v_0$ is adjacent to $v_1 = (v_0)\tau_j$, we see that each $\tau_j$ has even period, at least equal to the minimum possible girth 4. However, these even periods can be different. □

It follows at once from the Lemma 1 that $\rho_0, \rho_1, \rho_2$ and $\rho_3$ are involutions satisfying the relations (1), with $p_1 = p_3 = 3$ and $p_2 = q$. However, $\Gamma$ might not satisfy the intersection condition (2) and hence need not be a string $C$-group. Even so, beginning with $\Gamma$ and guided by Definition 2 and [17, 2E], one can still construct an abstract symmetric structure which resembles a polytope, namely one of Grünbaum's polystromata [14]. For our purposes, we may define a *polystroma* of rank $n$ to be a partially ordered set with a strictly monotone rank function $\{-1, 0, \ldots, n\}$, and with a unique least face $F_{-1}$ and unique greatest face $F_n$.

**Definition 4.** Suppose that $\mathcal{G}$ is a finite, connected, trivalent and 3-transitive graph equipped with the group $\Gamma = \Gamma(\mathcal{G}) = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ described in Definition 3. Then the *polystroma* $\mathcal{P} = \mathcal{P}(\mathcal{G})$ based on $\mathcal{G}$ is the partially ordered set constructed from $\Gamma$ as in Definition 2.

In this case, $\Gamma$ acts as a group of automorphisms of $\mathcal{P}$ and is even transitive on chains of each given type $K \subseteq \{-1, \ldots, 4\}$. (By this we mean a chain whose faces have exactly the ranks in $K$.) In particular, $\mathcal{P}$ is *flag transitive*, and all sections of $\mathcal{P}$ of each fixed type are isomorphic (cf. [17, 2E8]). Again $\mathcal{P}$ is self-dual. However, it can happen that the homogeneity property fails badly. In short, $\mathcal{P}$ need not be a polytope.

As we have already noted in Examples 1 and 2, when $\mathcal{G}$ is one of Foster's graphs **6**, **18** or **20B**, the polystroma $\mathcal{P}(\mathcal{G})$ really is a polytope. For another interesting case, see Example 3 below. Normally, however, we would expect polytopality to fail; see Example 4 below. Let's take stock of the general situation:

**Theorem 3.** *Let $\mathcal{G}$ be a finite, connected, trivalent and* 3*-transitive graph. Suppose that the group $\Gamma = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ constructed in Definition* 3 *also satisfies the intersection condition* (2) *(so that $\Gamma$ is a $C$-group). Then $\Gamma$ is the automorphism group of a self-dual, regular polytope $\mathcal{P}$ of type* $\{3, q, 3\}$. *Furthermore,*

(a) *if $\mathcal{G}$ is bipartite, then $\Gamma$ has index* 2 *in* $\mathrm{Aut}(\mathcal{G}) \simeq D(\mathcal{P})$, *the extended group for the polytope* $\mathcal{P}$*; and $\mathcal{G}$ is isomorphic to the medial layer graph for $\mathcal{P}$.*
(b) *if $\mathcal{G}$ is non-bipartite, then $\Gamma = \mathrm{Aut}(\mathcal{G}) \simeq \mathrm{Aut}(\mathcal{P})$; and the medial layer graph of $\mathcal{P}$ is the canonical double covering of $\mathcal{G}$.*

**Proof.** By (2) and Lemma 1, $\Gamma$ is indeed the automorphism group of a regular 4-polytope $\mathcal{P}$ of type $\{3, q, 3\}$, whose faces $\Gamma_j\varphi$ are described in Definition 2. With this description of $\mathcal{P}$, it follows immediately from Lemma 1(a) that the mapping

$$\begin{aligned} &\Delta : \mathcal{P} \to \mathcal{P} \\ &\Gamma_j\varphi \mapsto \Gamma_{3-j}\delta\varphi\delta, \qquad (\varphi \in \Gamma), \end{aligned} \tag{7}$$

is a well-defined polarity. Thus the medial graph $\mathcal{H}$ of $\mathcal{P}$ is trivalent, bipartite and 3-transitive (Theorem 2).

It is similarly easy to check that the mapping

$$\begin{aligned} &\mathcal{H} \xrightarrow{f} \mathcal{G} \\ &\Gamma_1\alpha \mapsto (v_1)\alpha, \qquad \alpha \in \Gamma \\ &\Gamma_2\alpha \mapsto (v_2)\alpha, \qquad \alpha \in \Gamma \end{aligned}$$

is well-defined, onto and preserves adjacency.

Suppose now that $\mathcal{G}$ has $N$ vertices. Then by Theorem 1(a) we have $|\mathrm{Aut}(\mathcal{G})| = 12N$, and $|\Gamma| = 12N/k$, where $k = 1$ or 2 by Lemma 1(b). Recall that $|\Gamma_1| = |\Gamma_2| = 12$. Thus $\mathcal{H}$ has $2|\Gamma|/12 = 2N/k$ vertices. When $k = 2$, the graph homomorphism $f$ becomes an isomorphism, since $\mathcal{H}, \mathcal{G}$ are each trivalent with $N$ vertices.

When $\mathcal{G}$ is bipartite, we have $k = 2$ since all $\rho_j$ preserve the partition classes. Thus $\mathcal{H} \simeq \mathcal{G}$.

When $\mathcal{G}$ is non-bipartite, we have $k = 1$, since $\mathcal{H}$ (being bipartite) cannot be isomorphic to $\mathcal{G}$. In fact, it is easy to check that $\mathcal{H}$ is the canonical double covering of $\mathcal{G}$. (See [3, 19a] for a general description of this covering.) □

**Example 3.** Graphs **10** and **20B** in the Census.

It is interesting to apply part (b) of the theorem to the non-bipartite Petersen graph $\mathcal{G}$ (**10** in the *Census*). From this graph our construction yields the regular 4-simplex $\mathcal{P} = \{3, 3, 3\}$; as we have seen in Example 1, the medial graph $\mathcal{H}$ of $\mathcal{P}$ is graph **20B**, which is indeed the canonical double covering of the Petersen graph. □

We next observe that when the intersection condition (2) does fail for the polystroma $\mathcal{P}$ constructed above, it must do so in a subtle way.

**Theorem 4.** *Suppose $\Gamma := \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ is constructed as above for the finite, connected, trivalent* 3*-transitive graph $\mathcal{G}$. Then*

(a) *$\Gamma_0 = \langle \rho_1, \rho_2, \rho_3 \rangle$ and $\Gamma_3 = \langle \rho_0, \rho_1, \rho_2 \rangle$ are C-groups.*
(b) *$\Gamma$ is a C-group if and only if*

$$\Gamma_0 \cap \Gamma_3 = \langle \rho_1, \rho_2 \rangle.$$

**Proof.** Since $\delta\, \Gamma_3\, \delta = \Gamma_0$, we need only consider $\Gamma_3$. By considering the action of the $\rho_j$'s on the vertices displayed in Fig. 8, it is easy to check that $\rho_0, \rho_1, \rho_2, \rho_3$ are distinct involutions.

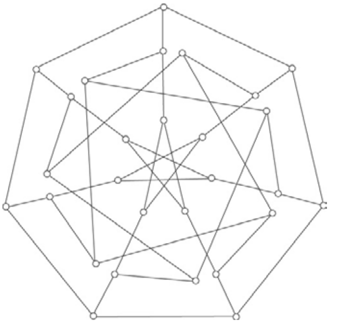

Fig. 9. The Coxeter graph — **28** in the *Census*.

Thus $\langle \rho_0, \rho_1 \rangle$ is dihedral of order 6 and $\langle \rho_1, \rho_2 \rangle$ is dihedral of some order $2q$, so that each is a $C$-group. To show that $\Gamma_3$ is a $C$-group, it suffices to check that

$$\langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle = \langle \rho_1 \rangle$$

([17, 2E16]).

But if this equality fails, then by the maximality of $\langle \rho_1 \rangle$ in $\langle \rho_0, \rho_1 \rangle \simeq \mathbb{S}_3$, we obtain $\rho_0 \in \langle \rho_1, \rho_2 \rangle$ (cf. [17, 11A10]). Thus, $\rho_0 = (\rho_1\rho_2)^m$ or $\rho_1(\rho_1\rho_2)^m$ for some $m$. In the first case, conjugation by $\rho_1$ gives

$$\rho_1\rho_0\rho_1 = (\rho_1\rho_2)^{-m} = \rho_0,$$

so that $(\rho_0\rho_1)^2 = e$, a contradiction. In the latter case, conjugation by $\rho_2$ similarly gives

$$\rho_1\rho_0\rho_1 = \rho_2\rho_1\rho_2.$$

From this it follows that $(v_2)\rho_2\rho_1\rho_2 = v_2$, hence $(v_0)\rho_1 = v_0$. Thus $v_3 = (v_1)\rho_1\rho_3$ is adjacent to $(v_0)\rho_1\rho_3 = (v_0)\rho_3 = w_1$, so that $(v_3)\rho_2 = v_3$. Then

$$v_3 = (v_3)\rho_1\rho_0\rho_1 = (v_3)\rho_2\rho_1\rho_2 = (v_3)\rho_1\rho_2 = v_1,$$

a contradiction. Hence, $\Gamma_0, \Gamma_3$ are $C$-groups; part (b) follows from [17, 2E16]. □

For any finite, connected 3-transitive trivalent graph (bipartite or not), Theorem 4 provides a peculiar construction for a regular 3-polytope (i.e. regular map) $\mathcal{M}$ of type $\{3, q\}$ (or $\{q, 3\}$). Of course, although $\Gamma_0$ is a $C$-group, the intersection condition may still fail for $\Gamma$ itself. In such cases there is little hope of faithfully realizing the map $\mathcal{M}$ as a section in the non-polytopal polystroma $\mathcal{P}$ of rank 4.

**Example 4.** The Coxeter graph (**28** in the Census).

Although it is 3-transitive, the well-known Coxeter graph $\mathcal{G}$ displayed in Fig. 9 is not bipartite and hence clearly is not the medial graph of a polytope. (See [9] for a detailed discussion of this very interesting graph.) In fact, even its 3-transitive and bipartite double cover $\mathcal{H}$ (**56C** in the *Census*) is not the medial graph of a regular polytope.

From [9] we note that $\mathrm{Aut}(\mathcal{G}) \simeq PGL(2, 7)$ is generated by the linear fractional transformations $\rho_0 := z \mapsto 1/z$, $\rho_1 := z \mapsto 1 - z$ and $\rho_2 := z \mapsto -z$ over $\mathbb{Z}_7$. Here $\rho_3 = (\rho_2\rho_1)^3\rho_0(\rho_2\rho_1)^{-3}$ is redundant, confirming that the group $\Gamma = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ is not polytopal. Thus $\rho_3 \in \Gamma_3$ and dually $\rho_0 \in \Gamma_0$. As a result, the polystroma $\mathcal{P}$ has a unique 0-face and unique 3-face.

Nevertheless, by Theorem 4(a), $\mathrm{Aut}(\mathcal{G})$ is the group of a regular map $\mathcal{M}$, namely Klein's map $\{7,3\}_8$. (See [10], in particular Table 8 and the references in Section 8.6.) $\square$

## 5. Chiral 4-polytopes of type {3, $q$, 3} and their medial layer graphs

For any regular $n$-polytope $\mathcal{P}$, the *rotations* $\sigma_j := \rho_{j-1}\rho_j$, $1 \leq j \leq n-1$, generate an interesting subgroup $\mathrm{Aut}(\mathcal{P})^+$ having index 1 or 2 in $\mathrm{Aut}(\mathcal{P})$. In the latter case, $\mathcal{P}$ is said to be *directly regular*, and certain properties of the $\sigma_j$ lead, in a natural way, to a parallel theory of *chiral* polytopes (see [21,22] for details).

So suppose that $\mathcal{P}$ has rank $n \geq 3$. Then $\mathcal{P}$ is *chiral* if it is not regular, but if for some *base flag* $\Phi := \{F_{-1}, F_0, \ldots, F_n\}$ there exist automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ of $\mathcal{P}$ such that $\sigma_j$ fixes all faces in $\Phi \setminus \{F_{j-1}, F_j\}$ and cyclically permutes consecutive $j$-faces of $\mathcal{P}$ in the rank 2 section $F_{j+1}/F_{j-2}$ of $\mathcal{P}$. The $\sigma_j$ can then be chosen so that if $F'_j$ denotes the $j$-face of $\mathcal{P}$ with $F_{j-1} < F'_j < F_{j+1}$ and $F'_j \neq F_j$, then $F_j\sigma_j = F'_j$ for $j = 1, \ldots, n-1$. The automorphism group of $\mathcal{P}$ now has two flag orbits, with adjacent flags always in different orbits.

Again, for brevity we now suppose that $\mathcal{P}$ is chiral with rank $n = 4$. Then the automorphisms $\sigma_1, \sigma_2, \sigma_3$ generate $\mathrm{Aut}(\mathcal{P})$ and satisfy at least the relations

$$\begin{aligned} &\sigma_1^{p_1} = \sigma_2^{p_2} = \sigma_3^{p_3} = e \\ &(\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = e, \end{aligned} \tag{8}$$

for some $2 \leq p_1, p_2, p_3 \leq \infty$. Once more $\mathcal{P}$ is equivelar of type $\{p_1, p_2, p_3\}$. Much as in the regular case, the specified generators satisfy an intersection condition:

$$\begin{aligned} &\langle\sigma_1\rangle \cap \langle\sigma_2\rangle = e = \langle\sigma_2\rangle \cap \langle\sigma_3\rangle, \\ &\langle\sigma_1, \sigma_2\rangle \cap \langle\sigma_2, \sigma_3\rangle = \langle\sigma_2\rangle. \end{aligned} \tag{9}$$

Each (isomorphism type of) chiral polytope gives rise to two *enantiomorphic* chiral polytopes: if one of them is associated with the base flag $\Phi$, the other is associated with an adjacent flag, say $\Phi^0 := (\Phi \setminus \{F_0\}) \cup \{F'_0\}$. As a result of this change, $\sigma_1, \sigma_2, \sigma_3$ are replaced by new generators $\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3$ (cf. [22, Section 3]).

Conversely, if a group $\Lambda = \langle\sigma_1, \sigma_2, \sigma_3\rangle$ satisfies (8) and (9), then there exists a chiral or directly regular 4-polytope $\mathcal{P} = \mathrm{Pol}(\Lambda)$ of type $\{p_1, p_2, p_3\}$. The construction is similar to that in the regular case (Definition 2), now taking $\Lambda_0 := \langle\sigma_2, \sigma_3\rangle$, $\Lambda_1 := \langle\sigma_1\sigma_2, \sigma_3\rangle$, $\Lambda_2 := \langle\sigma_1, \sigma_2\sigma_3\rangle$, $\Lambda_3 := \langle\sigma_1, \sigma_2\rangle$. We refer to [21] for details. Once more the polytope and group are recaptured in this correspondence, since $\Lambda \simeq \mathrm{Aut}(\mathrm{Pol}(\Lambda))$ (or $\mathrm{Aut}(\mathrm{Pol}(\Lambda))^+$ in the directly regular case), and $\mathcal{P} \simeq \mathrm{Pol}(\mathrm{Aut}(\mathcal{P}))$ (or $\mathrm{Pol}(\mathrm{Aut}(\mathcal{P})^+)$). The directly regular case occurs if and only if $\Lambda$ admits an involutory automorphism $\rho$ such that $(\sigma_1)\rho = \sigma_1^{-1}$, $(\sigma_2)\rho = \sigma_1^2\sigma_2$ and $(\sigma_3)\rho = \sigma_3$.

When $\mathcal{P}$ is chiral and self-dual, we encounter two subtly different cases (see [22, Section 3] and [15]).

**Case** $+$: $\mathcal{P}$ is *properly* self-dual. Here $\mathcal{P}$ admits a polarity (involutory duality) $\delta$ which reverses the base flag $\Phi$ and so preserves the two flag orbits. Consequently, in $D(\mathcal{P})$ we find that

$$\begin{aligned} &\delta^2 = e \\ &\delta\sigma_1\delta = \sigma_3^{-1} \\ &\delta\sigma_2\delta = \sigma_2^{-1} \\ &\delta\sigma_3\delta = \sigma_1^{-1}. \end{aligned} \tag{10}$$

Otherwise we have

**Case** $-$: $\mathcal{P}$ is *improperly* self-dual. Here there exists a duality $\delta$ which exchanges the two flag orbits. In fact, we may choose $\delta$ so that it maps $\Phi$ to $\Phi^0$ (order reversed). Then $\delta$ has period 4 and satisfies the relations

$$\begin{aligned}
&\delta^2 = \sigma_1\sigma_2\sigma_3\\
&\delta^{-1}\sigma_1\delta = \sigma_3^{-1}\\
&\delta^{-1}\sigma_2\delta = \sigma_1\sigma_2\sigma_1^{-1}\\
&\delta^{-1}\sigma_3\delta = \sigma_1
\end{aligned} \tag{11}$$

in $D(\mathcal{P})$.

Suppose now that $\mathcal{P}$ is a chiral polytope of type $\{3, q, 3\}$, and let $\mathcal{G}$ be its medial graph. We may again label the vertices as indicated in Fig. 4, so that $v_1 := F_1, v_2 := F_2$ and $(v_j)\sigma_2^k = v_{j-2k}$, still taking subscripts mod $2q$. For reference, we note that

$$\begin{aligned}
&\sigma_1 = (v_2)(v_1v_3w_2)(v_0v_4*)\ldots,\\
&\sigma_3 = (v_1)(v_0v_2w_1)(v_{-1}v_3*)\ldots.
\end{aligned}$$

**Theorem 5.** *Suppose $\mathcal{P}$ is a finite, chiral self-dual polytope of type* $\{3, q, 3\}$. *Then the medial layer graph $\mathcal{G}$ is 2-transitive and* $\mathrm{Aut}(\mathcal{G}) \simeq D(\mathcal{P})$. *Furthermore, $\mathcal{G}$ is of type* $2^+$ *(resp.* $2^-$*) if and only if $\mathcal{P}$ is properly (resp. improperly) self-dual.*

**Proof.** Each incident 1-face and 2-face in $\mathcal{P}$ can be extended to a flag in either chirality orbit. Because $\mathcal{P}$ is self-dual, $\mathrm{Aut}(\mathcal{G})$ is thus transitive on 1-arcs. But $\sigma_1\sigma_2\sigma_3$ maps $[v_2, v_1, v_0]$ to $[v_2, v_1, w_1]$ so that $\mathrm{Aut}(\mathcal{G})$ is transitive on $t$-arcs, for some $t \geq 2$. We next exclude the types $3^+$, $4^+$, $4^-$, $5^+$ (Theorem 1(d)). As in the proof of Theorem 2, we exploit the universal relations satisfied by $\mathrm{Aut}(\mathcal{G})$ in each instance [6, Section 1]. A detailed explanation in just one case will give the sense of the argument.

So suppose $\mathcal{G}$ is of type $3^+$. Then $\mathrm{Aut}(\mathcal{G})$ is generated by $h, a, p$, which with the redundant element $q := apa$ satisfy

$$\begin{aligned}
&h^3 = p^2 = a^2 = q^2 = (hq)^2 = e\\
&ph = hp, \qquad pq = qp.
\end{aligned} \tag{12}$$

Once more following [6, Theorem 1.3], we identify vertices in $\mathcal{G}$ with right cosets of the subgroup $H := \langle h, p, q\rangle$ (see Fig. 5). Since $\mathcal{G}$ is supposed to be 3-transitive, we may thus let

$$v_0 := Hah^{-1}, \quad v_1 := H, \quad v_2 := Ha, \quad v_3 := Haha.$$

Since $\sigma_3$ stabilizes $v_1$, we can identify $\sigma_3$ with an element of period 3 in $H$. But $H \simeq \mathbb{D}_{12}$, since $(hpq)^2 = q^2 = (hp)^6 = e$, so that only $h, h^{-1}$ have period 3 (cf. Theorem 1(c)). It is easy now to see that we are forced to take $\sigma_3 = h$ and similarly $\sigma_1 = aha$. Next consider

$$\alpha := \sigma_1\sigma_2 = (v_1)(v_0v_2)(w_0v_3)(w_1)(w_2v_{-1})\cdots.$$

Thus $\alpha$ is one of seven involutions in $H$, fixes $Hah$ and swaps $Ha$ with $Hah^{-1}$. Considering all possibilities, we find that

$$\sigma_2 = \sigma_1^{-1}\alpha = ah^2aqh^2p^m,$$

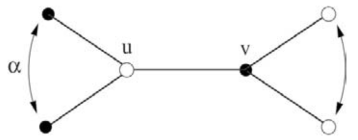


Fig. 10. The unique involution which fixes an arc in a 2-transitive graph.

for $m = 0$ or 1. But then the requirement that $(\sigma_2\sigma_3)^2 = e$ forces $m = 1$. However, now the involution $\rho := p$ satisfies

$$\rho\sigma_1\rho = \sigma_1^{-1}, \quad \rho\sigma_2\rho = \sigma_1^2\sigma_2, \quad \rho\sigma_3\rho = \sigma_3.$$

Thus, by our earlier remarks, $\mathcal{P}$ is directly regular, rather than chiral, a contradiction.

The arguments for types $4^+$, $4^-$, $5^+$ are similar though rather more involved. In each case, it is not even possible to achieve $(\sigma_2\sigma_3)^2 = e$.

We now know that $\mathcal{G}$ is 2-transitive. If the polytope $\mathcal{P}$ is properly self-dual, then the polarity

$$\delta = (v_1v_2)(v_0v_3)\ldots$$

gives shunts $\tau_1 := \alpha\delta = (v_0v_1v_2v_3\ldots)\ldots$, and $\tau_2 := \tau_1\sigma_1\sigma_2\sigma_3 = (v_0v_1v_2w_2\ldots)\ldots$. Thus $\alpha\tau_1\alpha = \tau_1^{-1}$, so that $\mathcal{G}$ has type $2^+$.

If $\mathcal{P}$ is improperly self-dual, we have the duality

$$\delta = (v_1v_2)(v_3v_0w_2w_1)\ldots.$$

Now redefine $\tau_2 := \alpha\delta$, $\tau_1 := \tau_1\sigma_1\sigma_2\sigma_3$. Thus $\alpha\tau_2\alpha = \tau_1^{-1}$ and $\mathcal{G}$ has type $2^-$. □

**Example 5.** A family of 2-transitive graphs including **112A** in the *Census*.

In [18, Theorem 6.1] we described chiral or directly regular polytopes $\mathcal{Q}_d$ of type $\{3, 6, 3\}$ and parametrized by Eisenstein integers $d = a + b\omega$ ($\omega = e^{2\pi i/3}$; $a, b \in \mathbb{Z}$). In particular, $\mathcal{Q}_d$ is chiral and properly self-dual when $q = a^2 - ab + b^2$ is a rational prime $\equiv 1 (\mathrm{mod}\ 3)$. From this we obtain a graph $\mathcal{G}_q$ of type $2^+$ and with $q(q^2 - 1)/3$ vertices. For example, when $d = 1 + 3\omega$, so that $q = 7$, we get Foster's graph **112A**.

On the other hand, when $d = 1 + 4\omega$, so that $q = 13$, we get the graph **728D**, newly described in [7]. Although most of our examples can be readily checked by hand, here we employ the presentation of $\mathrm{Aut}(\mathcal{G})$ provided in [7]; the calculations then are quite easy using *GAP* ([13]), along with the subsidiary package *GRAPE* ([16,23]). □

## 6. From 2-transitive graphs to chiral polytopes

Finally, in the context of chiral polytopes, we investigate a converse construction: given a finite, connected, trivalent and 2-transitive graph $\mathcal{G}$, can we reconstruct a chiral and self-dual polytope $\mathcal{P}$ whose medial graph is $\mathcal{G}$? With some adjustments, we proceed as in Section 3. First, we observe that given any arc $[u, v]$ in a 2-transitive $\mathcal{G}$, there is a unique involutory automorphism $\alpha$ fixing both $u, v$ (see Fig. 10).

As the starting point for our construction, we choose a particular 2-arc $[v_0, v_1, v_2]$ in $\mathcal{G}$, with adjacent vertices still labelled as in Fig. 8. Motivated by our discussion of $\mathrm{Aut}(\mathcal{P})$ in the chiral case, and noting the action of $\alpha = \sigma_1\sigma_2$ in the proof of Theorem 5, we make the following

**Definition 5.** Suppose that $\mathcal{G}$ is a finite, connected, trivalent and 2-transitive graph. With the basic 2-arc $[v_0, v_1, v_2]$ and its neighbours labelled as above, let $\alpha_1$ be the unique involutory automorphism fixing the vertices $v_2, w_2$. Similarly define $\alpha_2$ fixing $v_1, v_2$ and $\alpha_3$ fixing $v_1, w_1$; and let $\Lambda := \langle \alpha_1, \alpha_2, \alpha_3 \rangle \subseteq \mathrm{Aut}(\mathcal{G})$. Finally set $\sigma_1 := \alpha_2\alpha_1,\ \sigma_2 := \alpha_1\alpha_2\alpha_3,\ \sigma_3 := \alpha_3\alpha_2$.

**Remark.** The configuration of $\alpha_j$'s described in Definition 5 is not quite unique up to conjugacy in $\mathrm{Aut}(\mathcal{G})$: we could replace $\alpha_1$ by $\tilde{\alpha}_1 = \alpha_2\alpha_1\alpha_2$ fixing $v_2, v_3$. The effect is to replace the $\sigma_j$'s by $\tilde{\sigma}_1 = \alpha_2\tilde{\alpha}_1 = \sigma_1^{-1},\ \tilde{\sigma}_2 = \tilde{\alpha}_1\alpha_2\alpha_3 = \sigma_1^2\sigma_2,\ \tilde{\sigma}_3 = \sigma_3$. (Compare the discussion of *enantiomorphism* in [22, Section 3].)

Before recording various properties of these automorphisms, we also let $\delta$ be the unique automorphism mapping arc $[v_1, v_2, v_3]$ to $[v_2, v_1, v_0]$; and set $\tau_1 := \alpha_3\delta$, $\tau_2 := \tau_1\alpha_2$.

**Lemma 2.** *Let $\Lambda = \Lambda(\mathcal{G})$ be the group constructed in Definition 5. Then*

(a) *We have $\alpha_j^2 = e$; $\alpha_1 = \sigma_2\sigma_3$; $\alpha_2 = \sigma_1\sigma_2\sigma_3$; $\alpha_3 = \sigma_1\sigma_2$; and $\Lambda = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$.*
(b) *$\tau_1, \tau_2$ are shunts generating* $\mathrm{Aut}(\mathcal{G})$.
(c) *$\Lambda$ has index $k \le 2$ in* $\mathrm{Aut}(\mathcal{G})$.
(d) *$\sigma_1^3 = \sigma_2^q = \sigma_3^3 = (\sigma_1\sigma_2)^2 = (\sigma_1\sigma_2\sigma_3)^2 = (\sigma_2\sigma_3)^2 = e$, for some $q \ge 2$. (The indicated periods are achieved.)*
(e) *$\delta$ transforms the $\sigma_j$ as in Eq. (10) or (11), according as $\mathcal{G}$ is of type $2^+$ or $2^-$.*

**Proof.** Verification is again routine since $\mathcal{G}$ is 2-transitive. For (c) note that $\delta^2 \in \langle \alpha_2 \rangle$. Finally, $q \ge 2$, since $\sigma_2 = \alpha_1\alpha_2\alpha_3$ does not fix $v_2$. $\square$

**Remark.** If $\mathcal{G}$ is bipartite, then $\delta \notin \Lambda$, so that $k = 2$ in part (c) of the lemma.

Given $\mathcal{G}$ we can thus construct a group $\Lambda = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ satisfying relations (8) but perhaps not the intersection condition (9). Next, guided by [21], we make the following

**Definition 6.** Suppose that $\mathcal{G}$ is a finite, connected, trivalent 2-transitive graph equipped with the group $\Lambda = \Lambda(\mathcal{G}) = \langle \sigma_1, \sigma_2, \sigma_2 \rangle$ from Definition 5. Then the polystroma $\mathcal{P} = \mathcal{P}(\mathcal{G})$ based on $\mathcal{G}$ is the partially ordered set defined from $\Lambda$ (i.e. by imitating the construction in [21, Section 5]).

As suggested earlier, the construction is somewhat similar to that by which a regular polytope can be built from a string $C$-group, complicated, however, by less commuting of the generators $\sigma_j$. Nevertheless, we do obtain a rank 4, self-dual polystroma $\mathcal{P}$ on which $\Lambda$ acts as a group of automorphisms. Now $\Lambda$ has at most two flag orbits and is transitive on (proper) chains of each type $K \subset \{0, 1, 2, 3\}$. Furthermore, all sections of fixed type in $\mathcal{P}$ are isomorphic. As in Theorem 3 we can say considerably more in the polytopal case:

**Theorem 6.** *Suppose that $\mathcal{G}$ is a finite, connected, trivalent and 2-transitive graph. If the group $\Lambda$ constructed in Definition 5 satisfies the intersection condition (9), then $\Lambda$ is the automorphism group (rotation group) for a self-dual chiral (or directly regular) polytope $\mathcal{P}$ of type $\{3, q, 3\}$. Further, in the chiral case, $\mathcal{P}$ is properly self-dual if $\mathcal{G}$ is of type $2^+$, improperly self-dual if $\mathcal{G}$ is of type $2^-$. Finally,*

(a) *if $\mathcal{G}$ is bipartite, then $\Lambda$ has index 2 in $\mathrm{Aut}(\mathcal{G}) \simeq D(\mathcal{P})$, the extended group for the polytope $\mathcal{P}$, and $\mathcal{G}$ is isomorphic to the medial layer graph for $\mathcal{P}$. Here $\mathcal{P}$ must be chiral.*
(b) *if $\mathcal{G}$ is non-bipartite, then $\Lambda = \mathrm{Aut}(\mathcal{G}) \simeq \mathrm{Aut}(\mathcal{P})$, and the medial layer graph of $\mathcal{P}$ is the canonical double covering of $\mathcal{G}$. In this case, $\mathcal{P}$ can be chiral or directly regular.*

**Proof.** Parts (a), (b) follow much as in the proof of Theorem 3, having first replaced $\Gamma, \Gamma_j$ by $\Lambda, \Lambda_j$. In particular, when $\mathcal{G}$ is bipartite, the medial graph $\mathcal{H} \simeq \mathcal{G}$ is 2-transitive, so that $\mathcal{P}$ must be chiral by Theorem 2. (Examples below show that direct regularity can occur in case (b).)

It remains to deal with duality in $\mathcal{P}$. First of all, when $\mathcal{G}$ is of type $2^+$, the mapping defined by (7), again with $\Lambda$ in place of $\Gamma$, defines a polarity of $\mathcal{P}$. When $\mathcal{G}$ is of type $2^-$, so that $\delta^2 = \sigma_1\sigma_2\sigma_3$, we instead define

$$\begin{aligned} &\Delta : \mathcal{P} \to \mathcal{P} \\ &\Lambda_j\varphi \mapsto \Lambda_{3-j}\delta^{-1}\varphi\delta, \qquad (j = 0, 1, 2; \varphi \in \Lambda); \\ &\Lambda_3\varphi \mapsto \Lambda_0\sigma_1\sigma_2\delta^{-1}\varphi\delta, \qquad (\varphi \in \Lambda). \end{aligned} \tag{13}$$

Then $\Delta$ is a well-defined duality on $\mathcal{P}$, which maps the base flag $(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$ to the 0-adjacent flag (reversed). Here $\Delta^2 = \sigma_1\sigma_2\sigma_3$, considered as a mapping in $\mathrm{Aut}(\mathcal{P})$. (If $\mathcal{P}$ is directly regular, as can occur in (b), both of these dualities can be defined.) □

To conclude, we verify an analogue of Theorem 4.

**Theorem 7.** *Suppose $\Lambda = \langle\sigma_1, \sigma_2, \sigma_3\rangle$ is constructed as in Definition 5 for the finite, connected, trivalent 2-transitive graph $\mathcal{G}$. Then*

(a) *$\Lambda_0 := \langle\sigma_2, \sigma_3\rangle$ and $\Lambda_3 := \langle\sigma_1, \sigma_2\rangle$ are the 'rotation groups' for directly regular or chiral 3-polytopes.*
(b) *$\Lambda$ is the group of a chiral 4-polytope if and only if $\Lambda_0 \cap \Lambda_3 = \langle\sigma_2\rangle$.*

**Proof.** Part (b) follows from (9) and part (a). By [21, Theorem 1] and the symmetry of part (a), we see that it is enough to show that $\langle\sigma_1\rangle \cap \langle\sigma_2\rangle = \langle e\rangle$. If this fails, then we may assume $\sigma_1 = \sigma_2^k$ for some integer $k$. Since then $\sigma_1$ commutes with $\sigma_2$, we get $e = (\sigma_1\sigma_2)^2 = \sigma_1^2\sigma_2^2$, so that $\sigma_1 = \sigma_2^2$. From this we quickly find that $\sigma_2 = (v_0v_2)(v_3v_1w_2)\ldots$, so that $v_0, v_2$ are each adjacent to $v_1, v_3, w_2$. Finally, $w_1 = (v_0)\alpha_2$ is then also adjacent to $v_1, v_3, w_2$, so that $\mathcal{G} \simeq K_{3,3}$, which is 3-transitive: contradiction. □

**Example 6.** The dodecahedral graph (**20A** in the *Census*).

The 1-skeleton $\mathcal{G}$ of the regular dodecahedron is clearly non-bipartite and 2-transitive. In fact, $\Lambda$ does satisfy (9); and $\mathcal{G}$ is covered by the 3-transitive graph **40** in the *Census*. The corresponding self-dual regular polytope is the universal polytope of type $\{\{3, 6\}_{(2,0)}, \{6, 3\}_{(2,0)}\}$, with just 5 vertices (and 5 facets); see [17, 11E10].

In much the same way, we find that the non-bipartite graph **56B**, which is the 1-skeleton of the map $\{7, 3\}_8$, lifts to the polytopal and 2-transitive graph **112A**, discussed in Example 5. □

**Example 7.** Graph **192C** in the *Census*.

For this 2-transitive bipartite graph $\mathcal{G}$ we have $|\mathrm{Aut}(\mathcal{G})| = 6 \cdot 192 = 1152$. The subgroup $\Lambda$ is polytopal and is the automorphism group for a new chiral polytope $\mathcal{P}$ of type $\{3, 8, 3\}$. Each of the 12 facets $\mathcal{F}$ in $\mathcal{P}$ is a *regular* 3-polytope of type $\{3, 8\}$. Viewed otherwise, $\mathcal{F}$ is a regular map on a surface of genus 2, with 16 triangular faces and 6 vertices; see [10, Section 8.8]. Of course, each vertex figure has type $\{8, 3\}$ dual to $\mathcal{F}$. In either case, the rotation group of order 48 is actually isomorphic to $GL(2, 3)$. □

**Example 8.** Graph **448C** from the expanded census in [7].

As yet we have no instance of an improperly self-dual polytope of type $\{3, q, 3\}$. The single relevant graph (of type $2^-$) in [7] has 448 vertices. However, this graph is not polytopal. Using GAP it is easy to check for this graph that the group $\Lambda$ described in Definition 5 does not satisfy (9). □

To summarize the key ideas in this paper, we first recall that every finite regular (or chiral) 4-polytope $\mathcal{P}$ of type $\{3, q, 3\}$ automatically yields a 3-transitive (or 2-transitive) trivalent medial layer graph $\mathcal{G}$. In the reverse direction we have natural procedures for constructing such 4-polytopes from suitable graphs in certain cases. It would be very interesting to characterize such *polytopal graphs* in some workable fashion.

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## References

[1] D. Archdeacon, The medial graph and voltage–current duality, Discrete Math. 104 (1992) 111–141.
[2] N. Biggs, Presentations for Cubic Graphs, in: Computational Group Theory, Academic Press, London, 1984, pp. 57–63.
[3] N. Biggs, Algebraic Graph Theory, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1993.
[4] F. Buekenhout, Foundations of incidence geometry, in: F. Buekenhout (Ed.), Handbook of Incidence Geometry, Elsevier Science, Amsterdam, 1995, pp. 63–105.
[5] I.Z. Bouwer, et al. (Eds.), The Foster Census: R.M. Foster's Census of Connected Symmetric Trivalent Graphs, The Charles Babbage Research Centre, Winnipeg, 1988.
[6] M. Conder, P. Lorimer, Automorphism groups of symmetric graphs of valency 3, J. Combin. Theory Ser. B 47 (1989) 60–72.
[7] M. Conder, P. Dobcsányi, Trivalent symmetric graphs on up to 768 vertices, J. Combin. Math. Combin. Comput. (in press).
[8] H.S.M. Coxeter, The edges and faces of a 4-dimensional polytope, Congr. Numer. 28 (1980) 309–334.
[9] H.S.M. Coxeter, My graph, Proc. London Math. Soc. (3) 46 (1983) 117–136.
[10] H.S.M. Coxeter, W.O.J. Moser, Generators and Relations for Discrete Groups, 3rd ed., Springer, New York, 1972.
[11] H.S.M. Coxeter, A.I. Weiss, Twisted honeycombs $\{3, 5, 3\}_t$ and their groups, Geom. Dedicata 17 (1984) 169–179.
[12] D. Djoković, G. Miller, Regular groups of automorphisms of cubic graphs, J. Combin. Theory Ser. B 29 (1980) 195–230.
[13] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.3. http://www.gap-system.org, 2002.
[14] B. Grünbaum, Regularity of graphs, complexes and designs, in: Problèmes combinatoires et théorie des graphes, Colloq. Internat. C.N.R.S. No. 260, Orsay, 1977, 191–197.
[15] I. Hubard, A.I. Weiss, Self-duality of Chiral Polytopes, J. Combin. Theory Ser. A 111 (2005) 128–136.
[16] B.D. McKay, The nauty package. http://cs.anu.edu.au/people/bdm/nauty/.
[17] P. McMullen, E. Schulte, Abstract Regular Polytopes, in: Encyclopedia of Mathematics and its Applications, vol. 92, Cambridge University Press, Cambridge, 2002.
[18] B. Monson, A.I. Weiss, Eisenstein integers and related $C$-groups, Geom. Dedicata 66 (1997) 99–117.
[19] B. Monson, A.I. Weiss, Polytopes, honeycombs, groups and graphs, in: C. Davis, E. Ellers (Eds.) The Coxeter Legacy — Reflections and Projections, vol. 46, Fields Institute Communications, American Mathematical Society, Providence, 2005 (in press).
[20] B. Monson, T. Pisanski, E. Schulte, A.I. Weiss, Semisymmetric Graphs from Polytopes (submitted for publication).
[21] E. Schulte, A.I. Weiss, Chiral polytopes, in: P. Gritzmann, B. Sturmfels (Eds.), Applied Geometry and Discrete Mathematics. The Victor Klee Festschrift, in: DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 4, 1991, pp. 493–516.
[22] E. Schulte, A.I. Weiss, Chirality and projective linear groups, Discrete Math. 131 (1994) 221–261.
[23] L.H. Soicher, The GRAPE package for GAP. http://www.maths.qmul.ac.uk/~leonard/grape/.
[24] W.T. Tutte, Connectivity in Graphs, University of Toronto Press, Toronto, 1966.
[25] A.I. Weiss, On trivalent graphs embedded in twisted honeycombs, in: Combinatorics'81 (Rome, 1981), in: North-Holland Math. Stud., vol. 78, North-Holland, Amsterdam, 1983, pp. 781–787.
[26] A.I. Weiss, An infinite graph of girth 12, Trans. Amer. Math. Soc. 283 (2) (1984) 575–588.